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Anti Fuzzy Congruence on Product Lattices

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Abstract

In this work, the concept of anti fuzzy congruence on lattice L is introduced. Also if K be another lattice, then the concept of anti fuzzy congruence on the product $L \times K$ is discussed. Finally it is proved for every anti fuzzy congruence relation μ on $L \times K$, the anti fuzzy congruences μ_L and μ_K can be defined on L and K respectively such that $\mu = \mu_L \times \mu_K$.

Keywords: Fuzzy set theory, Lattices and related structures, Congruence relations, Direct product, Isomorphisms.

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1. Introduction

Fuzzy set theory is a mathematical theory introduced by Zadeh [36] to deal with uncertain or vague notions, using values in the unitinterval $[0, 1]$ to indicate the specialist's uncertainty when evaluating the membership degree of an element to a given set. Lattice theory has been used to consider fuzzy logic in a more general framework. See, e.g., the works on L-fuzzy set theory [5], BL-algebras of Hajek [7]and Brouwerian lattices [34]. In the history of fuzzy mathematics, fuzzy relations were early considered to be useful in various applications, and have therefore been extensively investigated. For a contemporary general approach to fuzzy relations one should look in Belohlavek's book [1], and also to other general publications e.g., the books by Klir and Yuan [9] and Turunen [33]. Relational equations and applications are presented by Di Nola, Sessa, Pedrycz and Sanchezin [4], and some new approaches to fuzzy relations are given by Ignjatovic, Ciric and Bogdanovicin [2, 8]. Das [3] and Yijia [35] have introduced the concept of fuzzy congruences in the background of semigroups. In this paper we introduce anti fuzzy equivalence relation and anti fuzzy congruence on lattices. Also we investigate direct product of anti fuzzy congruences and prove that every anti fuzzy congruence on the product lattice $L \times K$ is of the form $\mu \times \nu$ where μ and ν are anti fuzzy congruences on L and K respectively. Also we consider conditions that the product of factor lattices L/μ and K/ν is isomorphic to the factor lattice $(L \times K)/(\mu \times \nu)$.

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2. Anti fuzzy congruences

Definition 2.1. (See [6]) Let P be a nonempty set. A partial order P is a binary relation $\leqslant$ on P such that, for all $x, y, z \in P$, the following conditions are hold:
(1) $x \leqslant x$ (reflexivity);
(2) $x \leqslant y$ and $y \leqslant x$ imply $x = y$ (antisymmetry);
(3) $x \leqslant y$ and $y \leqslant z$ imply $x = z$ (transivity).
A set P equipped with an order relation $\leqslant$ is said to be an ordered set (or partially ordered set or poset).

Definition 2.2. (See [6]) A partially ordered set in which every pair of elements has a join (or least upper bound) and a meet (or greatest lower bound) is called a lattice.

Definition 2.3. (See [6]) Let L and K be lattices. Then map $\varphi : L \to K$ is an isomorphism if φ is one-to-one, onto and if $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ and $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ for all $a, b \in L$.

Definition 2.4. (See [6]) Let L and K be lattices. Define
$\wedge : L \times K \to L \times K$ by $(l_1, k_1) \wedge (l_2, k_2) = (l_1 \wedge l_2, k_1 \wedge k_2)$ and $\vee : L \times K \to L \times K$ by $(l_1, k_1) \vee (l_2, k_2) = (l_1 \vee l_2, k_1 \vee k_2)$ for all $l_1, l_2 \in L$ and $k_1, k_2 \in K$. Then $L \times K$ will be a lattice called the direct product of L and K.

Definition 2.5. (See [10]) Let X be an arbitrary set. A fuzzy set of X, we mean a function from X into $[0, 1]$. A fuzzy binary relation on X is a fuzzy set defined on $X \times X$.

Definition 2.6. Let X be a non empty set and μ be a fuzzy binary relation on X
such that
(1) $\mu(x, x) = 0$;
(2) $\mu(x, y) = \mu(y, x)$;
(3) $\mu(x, z) \leqslant \inf_{y \in X} \max\{\mu(x, y), \mu(y, z)\}$
for all $x, y, z \in X$. Then μ is called an anti fuzzy equivalence relation.

Definition 2.7. Let μ be an anti fuzzy equivalence relation on X. The similarity class for each $x \in x$ is the fuzzy set μ_x on X, in which the membership grade of each element $y \in X$ is $\mu(x, y)$, i. e., $\mu_x(y) = \mu(x, y)$. Then the similarity class for an element x represents the degree to which all the members of X are similar to x.

Lemma 2.8. Let X be a non empty set and μ be an anti fuzzy equivalence relation on X. Then $\mu_x = \mu_y$ if and only if $\mu(x, y) = 0$ for all $x, y \in X$.

Proof. Let $x, y \in X$. If $\mu_x = \mu_y$, then $\mu_x(y) = \mu_y(y) = 0$ and then $\mu(x, y) = 0$.
Conversely, if $\mu(x, y) = 0$, then $\mu_x(y) = 0 = \mu_y(y)$ and so $\mu_x = \mu_y$. □

Definition 2.9. Let X be a lattice and μ be an anti fuzzy equivalence relation on X. Then μ is join compatible if

$$\mu(x_1 \vee x_2, y_1 \vee y_2) \leqslant \mu(x_1, y_1) \vee \mu(x_2, y_2)$$

and μ is meet compatible if

$$\mu(x_1 \wedge x_2, y_1 \wedge y_2) \leqslant \mu(x_1, y_1) \vee \mu(x_2, y_2)$$

for all x_1, x_2, y_1, y_2 in X. If μ is both join compatible and meet compatible, then μ is an anti fuzzy congruence on X. Denote by $AFC(X)$, the set of all anti fuzzy congruences on lattice X.

Example 2.10. The fuzzy binary relation μ defined on a lattice X by

$$\mu(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

is an anti fuzzy congruence on X for all $x, y \in X$.

Lemma 2.11. Let X be a lattice and μ be an anti fuzzy equivalence relation on X. Then
(1) μ is join compatible if and only if $\mu(x_1 \vee t, y_1 \vee t) \leqslant \mu(x_1, y_1)$,
(2) μ is meet compatible if and only if $\mu(x_1 \wedge t, y_1 \wedge t) \leqslant \mu(x_1, y_1)$,
for all x_1, y_1, t in X.

Proof. Let x_1, x_2, y_1, y_2, t in X.
(1) If μ is join compatible, then

$$\mu(x_1 \vee t, y_1 \vee t) \leqslant \mu(x_1, y_1) \vee \mu(t, t) = \mu(x_1, y_1) \vee 0 = \mu(x_1, y_1).$$

Conversely, let $\mu(x_1, y_1) \geqslant \mu(x_1 \vee t, y_1 \vee t)$ and $\mu(x_2, y_2) \geqslant \mu(x_2 \vee t, y_2 \vee t)$. Then

$$\mu(x_1, y_1) \vee \mu(x_2, y_2) \geqslant \mu(x_1 \vee t, y_1 \vee t) \vee \mu(x_2 \vee t, y_2 \vee t) \geqslant \mu(x_1, y_1) \vee \mu(x_2, y_2).$$

(2) The proof is similar as (1). □

3. Direct product of anti fuzzy congruences

Definition 3.1. Let L and K be sets, μ and ν be binary fuzzy relations on L and K respectively. Define the fuzzy relation $\mu \times \nu$ on $L \times K$ by

$$(\mu \times \nu)((l_1, k_1), (l_2, k_2)) = \mu(l_1, l_2) \vee \nu(k_1, k_2)$$

for all l_1, l_2 in L and k_1, k_2 in K.

Proposition 3.2. Let $\mu \in AFC(L)$ and $\nu \in AFC(K)$. Then $\mu \times \nu \in AFC(L \times K)$.

Proof. Let l_1, l_2, l_3 in L and k_1, k_2, k_3 in K. Then
(1)

$$(\mu \times \nu)((l_1, k_1), (l_1, k_1)) = \mu(l_1, l_1) \vee \nu(k_1, k_1) = 0 \vee 0 = 0.$$

(2)

$$\begin{aligned}(\mu \times \nu)((l_1, k_1), (l_2, k_2)) &= \mu(l_1, l_2) \vee \nu(k_1, k_2)\\ &= \mu(l_2, l_1) \vee \nu(k_2, k_1)\\ &= (\mu \times \nu)((l_2, k_2), (l_1, k_1)).\end{aligned}$$

(3)

$$\begin{aligned}(\mu \times \nu)((l_1, k_1), (l_3, k_3)) &= \mu(l_1, l_3) \vee \nu(k_1, k_3)\\ &\leqslant \inf_{l_2 \in L}\{\mu(l_1, l_2) \vee \mu(l_2, l_3)\} \vee \inf_{k_2 \in K}\{\nu(k_1, k_2) \vee \nu(k_2, k_3)\}\\ &= \inf_{(l_2, k_2) \in (L \times K)}\{\mu(l_1, l_2) \vee \mu(l_2, l_3) \vee \nu(k_1, k_2) \vee \nu(k_2, k_3)\}\\ &= \inf_{(l_2, k_2) \in (L \times K)}\{\mu(l_1, l_2) \vee \nu(k_1, k_2) \vee \mu(l_2, l_3) \vee \nu(k_2, k_3)\}\\ &= \inf_{(l_2, k_2) \in (L \times K)}\{(\mu \times \nu)((l_1, k_1), (l_2, k_2)) \vee (\mu \times \nu)((l_2, k_2), (l_3, k_3))\}\\ &= \inf_{(l_2, k_2) \in (L \times K)} \max\{(\mu \times \nu)((l_1, k_1), (l_2, k_2)), (\mu \times \nu)((l_2, k_2), (l_3, k_3))\}.\end{aligned}$$

Thus $\mu \times \nu$ will be an anti fuzzy equivalence relation on $L \times K$.
Now By using Lemma 2.11 we prove that $\mu \times \nu$ is meet and join compatible. Let $(t_1, t_2) \in L \times K$ then

$$\begin{aligned}(\mu\times\nu)((l_1,k_1)\vee(t_1,t_2),(l_2,k_2)\vee(t_1,t_2)) &= (\mu\times\nu)((l_1\vee t_1,k_1\vee t_2),(l_2\vee t_1,k_2\vee t_2))\\ &= \mu(l_1\vee t_1,l_2\vee t_1)\vee\nu(k_1\vee t_2,k_2\vee t_2)\\ &\geqslant \mu(l_1,l_2)\vee\nu(k_1,k_2)\\ &= (\mu\times\nu)((l_1,k_1),(l_2,k_2))\end{aligned}$$

and thus $\mu\times\nu$ is join compatible. Also

$$\begin{aligned}(\mu\times\nu)((l_1,k_1)\wedge(t_1,t_2),(l_2,k_2)\wedge(t_1,t_2)) &= (\mu\times\nu)((l_1\wedge t_1,k_1\wedge t_2),(l_2\wedge t_1,k_2\wedge t_2))\\ &= \mu(l_1\wedge t_1,l_2\wedge t_1)\vee\nu(k_1\wedge t_2,k_2\wedge t_2)\\ &\geqslant \mu(l_1,l_2)\vee\nu(k_1,k_2)\\ &= (\mu\times\nu)((l_1,k_1),(l_2,k_2))\end{aligned}$$

and then $\mu\times\nu$ is meet compatible. Therefore $\mu\times\nu\in\mathsf{AFC}(L\times K)$. □

Example 3.3. Let $\mu\in\mathsf{AFC}(X)$ as in Example 2.10. Then $\mu\times\mu$ is defined by

$$(\mu\times\mu)((x,y),(z,t)=\mu(x,z)\vee\mu(y,t)=\begin{cases}0 & \text{if } (x,y)=(z,t)\\ 1 & \text{otherwise}\end{cases}$$

is an anti fuzzy congruence on $X\times X$ for all $x,y,z,t\in X$.

Proposition 3.4. Let $\beta\in\mathsf{AFC}(L\times K)$. Then for all l_1,l_2 in L and k_1,k_2 in K we have the following statements.
(1) $\beta((l_1,k_1),(l_2,k_1))=\beta((l_1,k_2),(l_2,k_2))$.
(2) $\beta((l_1,k_1),(l_1,k_2))=\beta((l_2,k_1),(l_2,k_2))$.

Proof. Let l_1,l_2 in L and k_1,k_2 in K. Then

$$\begin{aligned}\beta((l_1,k_1),(l_2,k_1)) &\geqslant \beta((l_1,k_1)\vee(l_1\wedge l_2,k_2),(l_2,k_1)\vee(l_1\wedge l_2,k_2)) \quad \text{(by Lemma 2.11)}\\ &= \beta((l_1\vee l_1\wedge l_2,k_1\vee k_2),(l_2\vee l_1\wedge l_2,k_1\vee k_2))=\beta((l_1,k_1\vee k_2),(l_2,k_1\vee k_2))\\ &\geqslant \beta((l_1,k_1\vee k_2)\wedge(l_1\vee l_2,k_2),(l_2,k_1\vee k_2)\wedge(l_1\vee l_2,k_2)) \quad \text{(by Lemma 2.11)}\\ &= \beta((l_1\wedge l_1\vee l_2,k_1\vee k_2\wedge k_2),(l_2\wedge l_1\vee l_2,k_1\vee k_2\wedge k_2))\\ &= \beta((l_1,k_2),(l_2,k_2)).\end{aligned}$$

Similarly it can be proved that $\beta((l_1,k_1),(l_2,k_1))\leqslant\beta((l_1,k_2),(l_2,k_2))$ and then $\beta((l_1,k_1),(l_2,k_1))=\beta((l_1,k_2),(l_2,k_2))$.
(2) The proof is similar to (1). □

Now we prove the converse of Proposition 3.2. On the other hand every anti fuzzy congruence relation on $L\times K$ is of form $\mu\times\nu$ as Proposition 3.2.

Proposition 3.5. Let $\beta\in\mathsf{AFC}(L\times K)$. Define binary fuzzy relations β_L on L and β_K on K as:

$$\beta_L(l_1,l_2)=\beta((l_1,k_1),(l_2,k_1)) \qquad \text{and} \qquad \beta_K(k_1,k_2)=\beta((l_1,k_1),(l_1,k_2))$$

for all $l_1,l_2\in L$ and $k_1,k_2\in K$. Then $\beta=\beta_L\times\beta_K$.

Proof. From Proposition 3.4 we get that β_L and β_K are well defined. We prove that β_L is an anti fuzzy congruence relation on L. Let $l_1, l_2 \in L$ and $k_1, k_2 \in K$. Then
(1) $\beta_L(l_1, l_1) = \beta((l_1, k_1), (l_1, k_1)) = 0$.
(2) $\beta_L(l_1, l_2) = \beta((l_1, k_1), (l_2, k_1)) = \beta((l_2, k_1), (l_1, k_1)) = \beta_L(l_2, l_1)$.
(3)

$$\begin{aligned}
\beta_L(l_1, l_2) &= \beta((l_1, k_1), (l_2, k_1)) \\
&\leqslant \inf_{(l_3,k_3)\in(L\times K)} \{\beta((l_1, k_1), (l_3, k_3)) \vee \beta((l_3, k_3), (l_2, k_1))\} \\
&\leqslant \inf_{l_3\in L} \{\beta((l_1, k_1), (l_3, k_1)) \vee \beta((l_3, k_1), (l_2, k_1))\} \\
&= \inf_{l_3\in L} \{\beta_L((l_1, l_3) \vee \beta_L((l_3, l_2)\} \\
&= \inf_{l_3\in L} \max\{\beta_L((l_1, l_3), \beta_L((l_3, l_2)\}.
\end{aligned}$$

(4)

$$\begin{aligned}
\beta_L(l_1 \vee l_3, l_2 \vee l_3) &= \beta((l_1 \vee l_3, k_1), (l_2 \vee l_3, k_1)) \\
&= \beta((l_1, k_1) \vee (l_3, k_1), (l_2, k_1) \vee (l_3, k_1)) \\
&\leqslant \beta((l_1, k_1), (l_2, k_1)) \\
&= \beta_L(l_1, l_2).
\end{aligned}$$

(5)

$$\begin{aligned}
\beta_L(l_1 \wedge l_3, l_2 \wedge l_3) &= \beta((l_1 \wedge l_3, k_1), (l_2 \wedge l_3, k_1)) \\
&= \beta((l_1, k_1) \wedge (l_3, k_1), (l_2, k_1) \wedge (l_3, k_1)) \\
&\leqslant \beta((l_1, k_1), (l_2, k_1)) \\
&= \beta_L(l_1, l_2).
\end{aligned}$$

Now (1)-(5) show that $\beta_L \in AFC(L)$. In a similar way we can prove that $\beta_K \in AFC(K)$. Next we must show that $\beta = \beta_L \times \beta_K$.

$$\begin{aligned}
(\beta_L \times \beta_K)((l_1, k_1), (l_2, k_2)) &= \beta_L(l_1, l_2) \vee \beta_K(k_1, k_2) \\
&= \beta((l_1, k_3), (l_2, k_3)) \vee \beta((l_3, k_1), (l_3, k_2)) \\
&= \beta((l_1, k_1 \wedge k_2), (l_2, k_1 \wedge k_2)) \vee \beta((l_1 \wedge l_2, k_1), (l_1 \wedge l_2, k_2)) \text{(Proposition 3.4)} \\
&\geqslant \beta((l_1, k_1 \wedge k_2) \vee (l_1 \wedge l_2, k_1), (l_2, k_1 \wedge k_2) \vee (l_1 \wedge l_2, k_2))) \\
&= \beta((l_1 \vee l_1 \wedge l_2, k_1 \wedge k_2 \vee k_1), (l_2 \vee l_1 \wedge l_2, k_1 \wedge k_2 \vee k_2)) \\
&= \beta((l_1, k_1), (l_2, k_2)).
\end{aligned}$$

Thus $\beta_L \times \beta_K \geqslant \beta$. (†)
Now

$$\begin{aligned}
\beta((l_1, k_1), (l_2, k_2)) &\geqslant \beta((l_1, k_1) \wedge (l_1 \vee l_2, k_1 \wedge k_2)), (l_2, k_2) \wedge (l_1 \vee l_2, k_1 \wedge k_2)) \quad \text{(by Lemma 2.11)} \\
&= \beta((l_1 \wedge l_1 \vee l_2, k_1 \wedge k_1 \wedge k_2), (l_2 \wedge l_1 \vee l_2, k_2 \wedge k_1 \wedge k_2)) \\
&= \beta((l_1, k_1 \wedge k_2), (l_2, k_1 \wedge k_2)) \\
&= \beta_L(l_1, l_2).
\end{aligned}$$

Thus $\beta((l_1,k_1),(l_2,k_2)) \geqslant \beta_L(l_1,l_2)$. (a) Also

$$\begin{aligned}\beta((l_1,k_1),(l_2,k_2)) &\geqslant \beta((l_1,k_1)\wedge(l_1\wedge l_2,k_1\vee k_2),(l_2,k_2)\wedge(l_1\wedge l_2,k_1\vee k_2)) \quad \text{(by Lemma 2.11)}\\ &= \beta((l_1\wedge l_1\wedge l_2,k_1\wedge k_1\vee k_2),(l_2\wedge l_1\wedge l_2,k_2\wedge k_1\vee k_2))\\ &= \beta((l_1\wedge l_2,k_1),(l_1\wedge l_2,k_2)\\ &= \beta_K(k_1,k_2).\end{aligned}$$

Therefore $\beta((l_1,k_1),(l_2,k_2)) \geqslant \beta_K(k_1,k_2)$. (b)
Now from (a) and (b) we get that

$$\beta((l_1,k_1),(l_2,k_2)) \geqslant \beta_L(l_1,l_2)\vee\beta_K(k_1,k_2) = (\beta_L\times\beta_K)((l_1,k_1),(l_2,k_2))$$

and then $\beta \geqslant \beta_L\times\beta_K$. (‡)
Then by (†) and (‡) we obtain that $\beta = \beta_L\times\beta_K$. □

Example 3.6. Let $\mu\in AFC(X\times X)$ as in Example 3.3 such that

$$\mu((x,y),(z,t)) = \mu(x,z)\vee\mu(y,t) = \begin{cases}0 & \text{if } (x,y)=(z,t)\\ 1 & \text{otherwise.}\end{cases}$$

Define the fuzzy binary relation μ_1 on X by

$$\mu_1(x,y) = \mu((x,z),(y,z)) = \begin{cases}0 & \text{if } (x,z)=(y,z)\\ 1 & \text{otherwise}\end{cases} = \begin{cases}0 & \text{if } x=y\\ 1 & \text{otherwise}\end{cases}$$

and the fuzzy binary relation μ_2 on X by

$$\mu_2(z,t) = \mu((x,z),(x,t)) = \begin{cases}0 & \text{if } (x,z)=(x,t)\\ 1 & \text{otherwise}\end{cases} = \begin{cases}0 & \text{if } z=t\\ 1 & \text{otherwise}\end{cases}$$

and then $\mu=\mu_1\times\mu_2$.

Remark 3.7. In the Propositions 3.2 and 3.5 it is also proved that if $\mu\in AFC(L\times K)$, then corresponding to each $k\in K$ we can define $\mu_L\in AFC(L)$ and corresponding to each $l\in L$, we can define $\mu_K\in AFC(K)$ such that $\mu=\mu_L\times\mu_K$ where $\mu_L(l_1,l_2)=\mu((l_1,k),(l_2,k))$ and $\mu_K(k_1,k_2)=\mu((l,k_1),(l,k_2))$ for all $l_1,l_2\in L$ and $k_1,k_2\in K$.

Definition 3.8. Let μ be an anti fuzzy congruence on lattice X. Then μ is an anti fuzzy equivalence relation and determines similarity classes. Let X/μ denote the set of all similarity classes of X determined by the anti fuzzy congruence μ. Suppose $X/\mu=\{\mu_x \mid x\in X\}$ where $\mu_x: X\to[0,1]$ such that $\mu_x(y)=\mu(x,y)$ for all $y\in X$. Now define two binary operations $\veebar$ and $\barwedge$ on X/μ by $\mu_x\veebar\mu_y=\mu_{x\vee y}$ and $\mu_x\barwedge\mu_y=\mu_{x\wedge y}$ for all $x,y\in X$. Then X/μ together with the binary operations $\veebar$ and $\barwedge$ is a lattice, which we call the factor lattice of X corresponding to the anti fuzzy congruence μ on X.

Proposition 3.9. Let L,K,μ,ν and $\mu\times\nu$ be as in Proposition 3.2. Then the factor lattice $(L\times K)/(\mu\times\nu)$ corresponding to $\mu\times\nu$ is isomorphic to the product of the corresponding factor lattices L/μ and K/ν.

Proof. Let $L/\mu=\{\mu_l \mid l\in L\}$ and $K/\nu=\{\nu_k \mid k\in K\}$. Moreover let

$$(L\times K)/(\mu\times\nu)=\{(\mu\times\nu)_{(l,k)} \mid (l,k)\in L\times K\}$$

and define the map

$$\varphi: L/\mu\times K/\nu\to(L\times K)/(\mu\times\nu) \quad\text{by}\quad \varphi(\mu_l,\nu_k)=(\mu\times\nu)_{(l,k)}.$$

Let $l_1, l_2 \in L$ and $k_1, k_2 \in K$. First we show that φ is well defined. Let $(\mu_{l_1}, \nu_{k_1}) = (\mu_{l_2}, \nu_{k_2})$ then $\mu_{l_1} = \mu_{l_2}$ and $\nu_{k_1} = \nu_{k_2}$. From Lemma 2.8 we have that $\mu(l_1, l_2) = 0$ and $\nu(k_1, k_2) = 0$ and so $(\mu \times \nu)((l_1, k_1), (l_2, k_2)) = \mu(l_1, l_2) \vee \nu(k_1, k_2) = 0 \vee 0 = 0$. Now Lemma 2.8 conclude $(\mu \times \nu)_{(l_1,k_1)} = (\mu \times \nu)_{(l_2,k_2)}$. Next we prove that φ is one to one. If $(\mu\times\nu)_{(l_1,k_1)} = (\mu\times\nu)_{(l_2,k_2)}$, then $(\mu\times\nu)((l_1, k_1), (l_2, k_2)) = 0$ and so $\mu(l_1, l_2) \vee \nu(k_1, k_2) = 0$. Then $\mu(l_1, l_2) = 0 = \nu(k_1, k_2)$ and by Lemma 2.8 we obtain $\mu_{l_1} = \mu_{l_2}$ and $\nu_{k_1} = \nu_{k_2}$ and $(\mu_{l_1}, \nu_{k_1}) = (\mu_{l_2}, \nu_{k_2})$. It is clearly that φ is onto. Finally we prove that φ is a lattice homomorphism. Let $(\mu_{l_1}, \nu_{k_1}), (\mu_{l_2}, \nu_{k_2}) \in L/\mu \times K/\nu$ and $\underline{\vee}(\overline{\wedge})$ be the join(meet) in factor lattice. Then

$$\begin{aligned}
\varphi((\mu_{l_1}, \nu_{k_1}) \,\underline{\vee}\, (\mu_{l_2}, \nu_{k_2})) &= \varphi(\mu_{l_1} \,\underline{\vee}\, \mu_{l_2}, \nu_{k_1} \,\underline{\vee}\, \nu_{k_2})\\
&= \varphi(\mu_{l_1\vee l_2}, \nu_{k_1\vee k_2}) \quad \text{(by Definition 3.8)}\\
&= (\mu\times\nu)_{(l_1\vee l_2, k_1\vee k_2)}\\
&= (\mu\times\nu)_{(l_1,k_1)\vee(l_2\vee k_2)}\\
&= (\mu\times\nu)_{(l_1,k_1)} \,\underline{\vee}\, (\mu\times\nu)_{(l_2,k_2)}\\
&= \varphi(\mu_{l_1}, \nu_{k_1}) \,\underline{\vee}\, \varphi(\mu_{l_2}, \nu_{k_2}).
\end{aligned}$$

Similarly

$$\varphi((\mu_{l_1}, \nu_{k_1}) \,\overline{\wedge}\, (\mu_{l_2}, \nu_{k_2})) = \varphi(\mu_{l_1}, \nu_{k_1}) \,\overline{\wedge}\, \varphi(\mu_{l_2}, \nu_{k_2}).$$

Therefore φ is a lattice homomorphism and proof is complete. □

4. Open problem

Norms were introduced in the framework of probabilistic metric spaces. However, they are widely applied in several other fields, e.g., in fuzzy set theory, fuzzy logic, and their applications. Now one can investigate norms over them and obtian some new results as author by using norms, investigated some properties of fuzzy algebraic structures [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

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